On the Mittag-Leffler Property

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Abstract: Let *C* be a category with strong monomorphic strong coimages, that is, every morphism *f* of *C* factors as $f = u \circ g$ so that *g* is a strong epimorphism and *u* is a strong monomorphism and this factorization is universal. We define the notion of strong Mittag-Leffler property in pro-*C*. We show that if $f : X \to Y$ is a level morphism in pro-*C* such that $p(Y)^{\beta}_{\alpha}$ is a strong epimorphism for all $\beta > \alpha$, then *X* has the strong Mittag-Leffler property provided *f* is an isomorphism. Also, if $f : X \to Y$ is a strong epimorphism of pro-*C* and *X* has the strong Mittag-Leffler property, we show that *Y* has the strong Mittag-Leffler property. Moreover, we show that this property is invariant of isomorphisms of pro-*C*.

Keywords: Pro-categories, strong Mittag-Leffler property, categories with strong monomorphic strong coimages. **MSC:** Primary 16B50.

1. INTRODUCTION

In [1], J. Dydak and F. R. Ruiz del Portal generalized the notion of Mittag-Leffler property to arbitrary balanced categories with epimorphic images. They obtained several results.

In [2], the author defined the notion of categories with strong monomorphic strong coimages. *C* is a category with strong monomorphic strong coimages if every morphism *f* of *C* factors as $f = u \circ g$ so that *g* is a strong epimorphism and *u* is a strong monomorphism and this factorization is universal among such factorization. In this paper, we define the notion of strong Mittag-Leffler property in pro-*C*. We show that if $f : X \to Y$ is a level morphism in pro-*C* such that $p(Y)_{\alpha}^{\beta}$ is a strong epimorphism for all $\beta > \alpha$, then *X* has the strong Mittag-Leffler property provided *f* is an isomorphism (Theorem 3.2). Also, if $f : X \to Y$ is a strong epimorphism of pro-*C* and *X* has the strong Mittag-Leffler property (Corollary 3.6). Moreover, we show that this property is invariant of isomorphisms of pro-*C* (Corollary 3.5).

2. PRELIMINARIES

First we recall some basic facts about pro-categories. The main reference is [3] and for more details see [4].

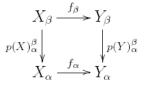
Let *C* be an arbitrary category. Loosely speaking, the pro-category pro-*C* of *C* is the universal category with inverse limits containing *C* as a full subcategory. An object of pro-*C* is an inverse system in *C*, denoted by $X = (X_{\alpha}, p_{\alpha}^{\beta}, A)$, consisting of a directed set *A*, called the *index*

set, of C objects X_{α} for each $\alpha \in A$, called the *terms* of X

and of *C* morphisms $p_{\alpha}^{\beta}: X_{\beta} \to X_{\alpha}$ for each related pair $\alpha < \beta$, called the *bonding morphisms* of *X*. A morphism of two objects $f: X = (X_{\alpha}, p_{\alpha}^{\beta}, A) \to Y = (Y_{\alpha'}, p_{\alpha'}^{\beta'}, A')$ consists of a function $\varphi: A' \to A$ and of morphisms $f_{\alpha'}: X_{\varphi(\alpha')} \to Y_{\alpha'}$ in *C* one for each $\alpha' \in A'$ such that whenever $\alpha' < \beta'$, then there is $\gamma \in A$, $\gamma > \varphi(\alpha'), \varphi(\beta')$ for which $f_{\alpha'} P_{\phi(\alpha')}^{\gamma} = P_{\alpha'}^{\beta'} f_{\beta'} P_{\phi(\beta')}^{\gamma}$. From now onward, the index set *A* of an object *X* of pro-*C* will be denoted by I(X) and the bonding morphisms by $P(X)_{\alpha}^{\beta}$ for each $\alpha < \beta$.

If *P* is an object of *C* and *X* is an object of pro-*C*, then a morphism $f: X \to P$ in pro-*C* is the direct limit of Mor (X_{α}, P) , $\alpha \in I(X)$ and so *f* can be represented by $g: X_{\alpha} \to P$. Note that the morphism from *X* to X_{α} represented by the identity $X_{\alpha} \to X_{\alpha}$ is called the *projection morphism* and denoted by $p(X)_{\alpha}$.

If X and Y are two objects in pro-C with identical index sets, then a morphism $f: X \to Y$ is called a *level morphism* if for each $\alpha < \beta$, the following diagram commutes.



Theorem 2.1. For any morphism $f: X \to Y$ of pro-*C* there exists a level morphism $f' = X' \to Y'$ and isomorphisms $i: X \to X', j: Y' \to Y$ such that $f = j \circ f' \circ i$ and I(X') is a

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