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The Bellman–Kalaba–Lakshmikantham Quasilinearization Method for Neumann Problems

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In this paper, we develop a generalized quasilinearization method for a nonlinear Neumann problem and obtain a sequence of approximate solutions converging monotonically and quadratically to a solution of the problem.

Key Words: quasilinearization; Neumann problem; quadratic convergence.

1. INTRODUCTION

The method of quasilinearization developed by Bellman and Kalaba [1] provides an explicit approach for obtaining approximate solutions to nonlinear differential equations and it gives point-wise lower estimates of the solution of the given problem provided the function involved is convex. Further, the sequence of approximate solutions converges monotonically and quadratically to the solution. Recently, this technique has received much attention after the publication of very interesting articles by Lakshmikantham *et al.* [4–8]. In these articles, the convexity assumption was

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surprisingly relaxed and the method was generalized and extended in several directions to make it applicable to a larger class of problems.

Shahzad and Vatsala [13, 14] and Shahzad and Sivasundaram [12] developed quasilinearization methods for second order boundary value problems. For a complete survey of the generalized quasilinearization method for nonlinear problems, see [9]. More recently, Nieto [10] presented a generalized quasilinearization technique for a nonlinear Dirichlet problem to obtain a sequence of approximate solutions converging quadratically to the solution of the problem.

In this paper, we discuss a second order ordinary nonlinear differential equation with Neumann boundary conditions—a problem in which the normal gradient of the unknown function is specified at each point of the boundary (Neumann Problem) and develop the method of quasilinearization for this problem.

2. PRELIMINARIES

We know that the Neumann boundary value problem

$$-\Psi''(t) = \lambda\Psi(t), \qquad t \in [0, \pi],$$
$$\Psi'(0) = \Psi'(\pi) = 0,$$

has a nontrivial solution if and only if $\lambda = m^2$ $(m = 0, 1, 2, 3, \dots)$. For $\lambda \neq m^2$ and $\zeta(t) \in C[0, \pi]$, the unique solution of the problem

$$-\Psi''(t) - \lambda\Psi(t) = \zeta(t), \qquad t \in [0, \pi],$$
$$\Psi'(0) = \Psi'(\pi) = 0,$$

is given by

$$\Psi(t) = \int_0^{\pi} G_\lambda(t, v)\zeta(v)\, dv,$$

where

$$G_\lambda(t, v) = \frac{1}{\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\,\pi)} \times \begin{cases} \cosh\left[\sqrt{-\lambda}\,(\pi - t)\right]\cosh\left[\sqrt{-\lambda}\,v\right], & 0 \le v \le t \le \pi \\ \cosh\left[\sqrt{-\lambda}\,t\right]\cosh\left[\sqrt{-\lambda}\,(\pi - v)\right], & 0 \le t \le v \le \pi \end{cases} \qquad (\lambda < 0)$$

$$G_\lambda(t, v) = \frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi)} \begin{cases} \cos\left[\sqrt{\lambda}\,(\pi - t)\right]\cos\left[\sqrt{\lambda}\,v\right], & 0 \le v \le t \le \pi \\ \cos\left[\sqrt{\lambda}\,t\right]\cos\left[\sqrt{\lambda}\,(\pi - v)\right], & 0 \le t \le v \le \pi \end{cases} \qquad (\lambda > 0).$$

Clearly, $G_\lambda \geq 0$ for $\lambda < 0$. Now, consider the following nonlinear problem

$$\begin{aligned} -x''(t) &= f(t, x(t)), \qquad t \in [0, \pi], \\ x'(0) &= x'(\pi) = 0, \end{aligned} \tag{2.1}$$

where $f : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is continuous. The problem (2.1) is equivalent to the integral equation

$$x(t) = x(0) - \int_0^t (t-s) f(s, x(s))\, ds$$

with

$$\int_0^\pi f(s, x(s))\, ds = 0.$$

We shall say that $\alpha \in C^2[[0, \pi], \mathbb{R}]$ is a lower solution of (2.1) if

$$\begin{aligned} -\alpha''(t) &\leq f(t, \alpha(t)), \qquad t \in [0, \pi], \\ \alpha'(0) &\geq 0, \qquad \alpha'(\pi) \leq 0. \end{aligned}$$

Similarly, we shall say that $\beta \in C^2[[0, \pi], \mathbb{R}]$ is an upper solution of (2.1) if

$$\begin{aligned} -\beta''(t) &\geq f(t, \beta(t)), \qquad t \in [0, \pi], \\ \beta'(0) &\leq 0, \qquad \beta'(\pi) \geq 0. \end{aligned}$$

2.1. LEMMA. *Assume that* $\alpha, \beta \in C^2[[0, \pi], \mathbb{R}]$ *are lower and upper solutions of* (2.1), *respectively*, *such that* $\alpha(t) \leq \beta(t)$ *for every* $t \in [0, \pi]$. *Then there exists a solution* $x(t)$ *of* (2.1) *such that* $\alpha(t) \leq x(t) \leq \beta(t)$ *for* $t \in [0, \pi]$.

We do not provide a proof of the lemma for it is similar to the proof of Theorem 2.1 of [11] (see also [3]).

It is worth mentioning that Lemma 2.1 is not valid for the natural definition of lower and upper solutions of (2.1), namely,

$$\begin{aligned} -\alpha''(t) &\leq f(t, \alpha(t)), \qquad t \in [0, \pi], \\ \alpha'(0) &\leq 0, \qquad \alpha'(\pi) \leq 0 \end{aligned}$$

and

$$\begin{aligned} -\beta''(t) &\geq f(t, \beta(t)), \qquad t \in [0, \pi], \\ \beta'(0) &\geq 0, \qquad \beta'(\pi) \geq 0. \end{aligned}$$

For example, consider the problem

$$\begin{aligned} -x''(t) &= 1, \qquad t \in [0, \pi], \\ x'(0) &= x'(\pi) = 0. \end{aligned}$$

It has no solution since $\int_0^\pi 1\, ds \neq 0$. Clearly, $\alpha(t) = \cos t$ is a lower solution and $\beta(t) = -\frac{1}{2}t^2 + \pi t + 1$ is an upper solution. Also, $\alpha \leq \beta$. However, there is no solution between α and β.

3. MAIN RESULT

3.1. THEOREM. *Assume that*

(A_1) $\alpha_o, \beta_o \in C^2[[0,\pi], \mathbb{R}]$ *are lower and upper solutions of* (2.1), *respectively, such that* $\alpha_o \leq \beta_o$ *on* $[0,\pi]$,

(A_2) $f \in C[\Omega, \mathbb{R}]$ *is such that* $f_x(t,x), f_{xx}(t,x)$ *exist and are continuous for every* $(t,x) \in \Omega$, *where*

$$\Omega = \{(t,x) \in [0,\pi] \times \mathbb{R} : \alpha_o(t) \leq x \leq \beta_o(t)\},$$

(A_3) $f_x(t,x) < 0$ *for every* $(t,x) \in \Omega$.

There there exists a monotone nondecreasing sequence $\{\alpha_n\}$ *which converges uniformly to a solution of* (2.1) *and the convergence is quadratic.*

Proof. Let $F(t,x) : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ be such that $F(t,x), F_x(t,x), F_{xx}(t,x)$ are continuous on $[0,\pi] \times \mathbb{R}$ and

$$F_{xx}(t,x) \geq 0, \qquad (t,x) \in [0,\pi] \times \mathbb{R}. \tag{3.1}$$

Motivated by Eloe and Zhang [2], take $\Phi(t,x) = F(t,x) - f(t,x)$ on $[0,\pi] \times \mathbb{R}$. In view of (3.1), we see that

$$F(t,x) \geq F(t,y) + F_x(t,y)(x-y)$$

for $x \geq y$ and hence

$$f(t,x) \geq f(t,y) + F_x(t,y)(x-y) - [\Phi(t,x) - \Phi(t,y)]. \tag{3.2}$$

Consider the Neumann problem

$$\begin{aligned} -u'' &= g(t,u;\alpha_o) \\ &= f(t,\alpha_o) + F_x(t,\alpha_o)(u-\alpha_o) - [\Phi(t,u) - \Phi(t,\alpha_o)] \qquad (3.3) \\ & u'(0) = u'(\pi) = 0. \end{aligned}$$

The inequality (3.2) and (A_1) imply

$$\begin{aligned} -\alpha_o'' &\leq f(t,\alpha_o) = g(t,\alpha_o;\alpha_o), \\ -\beta_o'' &\geq f(t,\beta_o) \\ &\geq f(t,\alpha_o) + F_x(t,\alpha_o)(\beta_o - \alpha_o) - [\Phi(t,\beta_o) - \Phi(t,\alpha_o)] \\ &= g(t,\beta_o;\alpha_o). \end{aligned}$$

By Lemma 2.1, there exists a solution α_1 of (3.3) such that $\alpha_o \leq \alpha_1 \leq \beta_o$ on $[0, \pi]$. Next, consider the Neumann problem

$$-u'' = g(t, u; \alpha_1) \tag{3.4}$$
$$u'(0) = u'(\pi) = 0.$$

Observe that

$$\begin{aligned}
-\alpha_1'' &= g(t, \alpha_1; \alpha_o) \\
&= f(t, \alpha_o) + F_x(t, \alpha_o)(\alpha_1 - \alpha_o) - [\Phi(t, \alpha_1) - \Phi(t, \alpha_o)] \\
&\leq f(t, \alpha_1) = g(t, \alpha_1; \alpha_1). \\
-\beta_o'' &\geq f(t, \beta_o) \\
&\geq f(t, \alpha_1) + F_x(t, \alpha_1)(\beta_o - \alpha_1) - [\Phi(t, \beta_o) - \Phi(t, \alpha_1)] \\
&= g(t, \beta_o; \alpha_1),
\end{aligned}$$

in view of (3.2). It now follows from Lemma 2.1 that there exists a solution α_2 such that $\alpha_1 \leq \alpha_2 \leq \beta_o$ on $[0, \pi]$. Consequently, we see that $\alpha_o \leq \alpha_1 \leq \alpha_2 \leq \beta_o$ on $[0, \pi]$. Continuing this process successively, we obtain

$$\alpha_o \leq \alpha_1 \leq \alpha_2 \cdots \leq \alpha_n \leq \beta_o$$

on $[0, \pi]$, where the elements of the monotone sequence $\{\alpha_n\}$ are the solutions of the problem

$$\begin{aligned}
-u'' &= g(t, u; \alpha_{n-1}) \\
&= f(t, \alpha_{n-1}) + F_x(t, \alpha_{n-1})(u - \alpha_{n-1}) - [\Phi(t, u) - \Phi(t, \alpha_{n-1})]
\end{aligned}$$
$$u'(0) = u'(\pi) = 0.$$

Since the sequence $\{\alpha_n\}$ is monotone, it follows that it has a pointwise limit x. Consider the following linear Neumann problem

$$-u'' = f_n(t), \tag{3.5}$$
$$u'(0) = u'(\pi) = 0,$$

where

$$f_n(t) = g(t, \alpha_n(t); \alpha_{n-1}(t)), \qquad t \in [0, \pi].$$

Since g is continuous on Ω, the sequence $\{f_n\}$ is bounded in $C[[0, \pi], \mathbb{R}]$. It is clear that

$$\lim_{n \to \infty} f_n(t) = f(t, x(t)), \qquad t \in [0, \pi].$$

But

$$\alpha_n(t) = \alpha_n(0) - \int_0^t (t-s) f_n(s)\, ds$$

and

$$\int_0^\pi f(s, \alpha_{n-1}(s))\, ds = \int_0^\pi \big(F_x(s, \alpha_{n-1}(s))[\alpha_n(s) - \alpha_{n-1}(s)]$$
$$-\Phi(s, \alpha_n(s)) + \Phi(s, \alpha_{n-1}(s))\big)\, ds.$$

Therefore $\{\alpha_n\}$ is bounded in $C^2[[0,\pi],\mathbb{R}]$ and so $\{\alpha_n\} \uparrow x$ uniformly on $[0,\pi]$. It further implies that

$$x(t) = x(0) - \int_0^t (t-s) f(s, x(s))\, ds, \qquad t \in [0,\pi]$$

and

$$\int_0^\pi f(s, x(s))\, ds = 0.$$

Hence x is a solution of (2.1).

To show the quadratic convergence, we set $p_n(t) = x(t) - \alpha_n(t)$. Then using the definition of α_n and the mean value theorem, we have

$$\begin{aligned}
-p_n''(t) &= f(t, x(t)) - g(t, \alpha_n(t); \alpha_{n-1}(t)) \\
&= f(t, x(t)) - f(t, \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))[\alpha_n(t) - \alpha_{n-1}(t)] \\
&\quad + [\Phi(t, \alpha_n(t)) - \Phi(t, \alpha_{n-1}(t))] \\
&= F(t, x, (t)) - F(t, \alpha_{n-1}(t)) \\
&\quad - F_x(t, \alpha_{n-1}(t))[\alpha_n(t) - \alpha_{n-1}(t)] \\
&\quad + [\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\
&= F_x(t, \xi)[x(t) - \alpha_{n-1}(t)] - F_x(t, \alpha_{n-1}(t))[\alpha_n(t) - \alpha_{n-1}(t)] \\
&\quad + [\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\
&= \big(F_x(t, \xi) - F_x(t, \alpha_{n-1}(t))\big)[x(t) - \alpha_{n-1}(t)] \\
&\quad + F_x(t, \alpha_{n-1}(t))[x(t) - \alpha_n(t)] \\
&\quad + [\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\
&= F_{xx}(t, \sigma)[\xi - \alpha_{n-1}(t)][x(t) - \alpha_{n-1}(t)] \\
&\quad + F_x(t, \alpha_{n-1}(t))[x(t) - \alpha_n(t)] - \Phi_x(t, \eta)[x(t) - \alpha_n(t)],
\end{aligned}$$

where

$$\alpha_{n-1}(t) \le \xi \le \sigma \le x(t), \qquad \alpha_n(t) \le \eta \le x(t).$$

Set

$$h_n(t) = F_x(t, \alpha_{n-1}(t)) - \Phi_x(t, \eta)$$

and

$$k_n(t) = F_{xx}(t, \sigma)[\xi - \alpha_{n-1}(t)][x(t) - \alpha_{n-1}(t)] - Mp_{n-1}^2(t),$$

where $0 \le F_{xx}(t, y) \le M$, $(t, y) \in \Omega$. Clearly $k_n(t) \le 0$. Since F_x is nondecreasing and $\alpha_{n-1}(t) \le \eta$, it follows by (A_3) that there exists $\lambda < 0$ and an integer N such that $h_n(t) \le \lambda$, $t \in [0, \pi]$ for $n \ge N$. Thus the error p_n satisfies the Neumann problem

$$-p_n''(t) - \lambda p_n(t) = [h_n(t) - \lambda] p_n(t) + Mp_{n-1}^2(t) + k_n(t),$$
$$p_n'(0) = p_n'(\pi) = 0.$$

This implies that

$$p_n(t) = \int_0^\pi G_\lambda(t, s)\{[h_n(s) - \lambda] p_n(s) + Mp_{n-1}^2(s) + k_n(s)\}\, ds,$$

and so

$$p_n(t) \le M\int_0^\pi G_\lambda(t, s) p_{n-1}^2(s)\, ds, \qquad n \ge N.$$

Hence there exists a constant $K > 0$ such that

$$\| p_n \| \le K \| p_{n-1} \|^2, \qquad n \ge N,$$

where $\|x\| = \max\{|x(t)| : t \in [0, \pi]\}$ is the usual uniform norm on $C[[0, \pi], \mathbb{R}]$.

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